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Local existence in $C_b^{0,1}$ and blow-up of the solutions of the Cauchy Problem for a quasilinear hyperbolic system with a singular source term*

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— Dedicated to Constantine Dafermos on his 60^{th} *birthday*

Abstract. In this paper we consider the Cauchy problem for the hyperbolic system

$$\begin{cases} a_t + (au)_x + \frac{2au}{x} = 0 \\ \\ u_t + \frac{1}{2}(a^2 + u^2)_x = 0 \end{cases} \qquad x > 0,\ t \geq 0$$

with null boundary conditions and we prove a local (in time) existence and uniqueness theorem in $C_b^{0,1}$ and, for a special class of initial data, a blow-up result.
Keywords: Hyperbolic quasilinear system, singular source term, blow-up of solutions.

1. Introduction and main results

We consider the Cauchy problem for the quasilinear hyperbolic system

$$\begin{cases} a_t + (au)_x + \frac{2au}{x} = 0 \\ \\ u_t + \frac{1}{2}(a^2 + u^2)_x = 0 \end{cases} \qquad x > 0,\ t \geq 0 \tag{1.1}$$

with the initial data

$$(a(x,0), u(x,0)) = (a_0(x),\ u_0(x)),\quad x > 0 \tag{1.2}$$

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The system (1.1) appears in the study of the radial symetric solutions in $\mathbb{R}^3 \times \mathbb{R}_+$ for a conservative system modeling the isentropic flow introduced by G.B. Whitham in [7, chap.9] where a is the sound speed and u is the radial velocity. If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $f(a,u) = (au, (1/2)(a^2+u^2))$, then two eingenvalues of ∇f are

$$\lambda_1 = u - a, \quad \lambda_2 = u + a \tag{1.3}$$

and so the strict hyperbolicity fails if $a = 0$, but the system is genuinely nonlinear with Riemann invariants

$$l = -u + a, \quad r = u + a \tag{1.4}$$

which satisfy the equivalent system (for classical solutions):

$$\begin{cases} r_t + rr_x + \frac{r^2-l^2}{2x} = 0 \\ \qquad\qquad\qquad\qquad x > 0,\ t \geq 0 \\ l_t - ll_x + \frac{r^2-l^2}{2x} = 0 \end{cases} \tag{1.5}$$

with initial data

$$(r(x,0), l(x,0)) = (r_0(x), l_0(x)), \ \ x > 0 \tag{1.6}$$

with $r_0 = u_0 + a_0$, $l_0 = -u_0 + a_0$.

In [1] and [2] we have studied, for a special class of initial data, the existence and uniqueness of weak entropy solutions of the Cauchy problem for system (1.1) verifying, in a certain sense, a null boundary condition. For this we have applied the vanishing viscosity method, the compensated compactness method of Tartar, Murat and DiPerna (cf. [3]) and, for the uniqueness under stronger assumptions, the Kruzkov's technique (cf. [4]). In this paper we deal with local (in time) $C^{0,1}$ solutions that are null at the boundary ($x = 0$) and, for commodity, we will work with the system (1.5). Let us introduce, for $T > 0$, the space

$$Y_T = \{v \in C_b^{0,1}([0, +\infty[\times[0,T] \mid v(0,t) = 0,\ \ 0 \leq t \leq T\} \tag{1.7}$$

where $C_b^{0,1}$ denotes the space of bounded Lipschitz continous functions, with the usual norm

$$\|v\|_{Y_T} = \|v\|_{L^\infty} + \|v_x\|_{L^\infty} + \|v_t\|_{L^\infty} \tag{1.8}$$

We will prove, by a standard fixed point method:

Theorem 1. *Assume* $r_0, l_0 \in (C_b^{0,1}([0,+\infty[))^2$ *and such that* $r_0(0) = l_0(0) = 0$. *Then, there exists* $T > 0$ *and a unique pair* $(r,l) \in Y_T \times Y_T$ *such that* (r,l) *is a solution of the Cauchy problem* (1.5), (1.6).

For each $x_0 > 0$ and $T > 0$ let us introduce

$$\Sigma_{x_0,T} = \{(x,t) \in]0,+\infty[\times [0,T] \mid x \geq X(t;x_0,0)\} \tag{1.9}$$

where $X(t;x_0,0)$ is the characteristic defined by

$$\frac{dX}{d\tau}(t;x_0,0) = r(X(t;x_0,0),t), \quad t \in [0,T], \ X(0;x_0,0) = x_0,$$

where (r,l) is the local solution of (1.5), (1.6) obtained in Theorem 1. We will prove the following regularity result:

Theorem 2. *Assume* $(r_0,l_0) \in (C_b^{1,1}([0,+\infty[))^2$, r_0 *and* l_0 *null at the origin and with compact support in* $[0,+\infty[$ *and* $u_0(x) \geq a_0(x) \geq 0$, $x \in \mathbb{R}_+$. *Then there exists a local solution* $(r,l) \in Y_T \times Y_T$ *of* (1.5), (1.6) *such that* $(r,l) \in (C_b^{1,1}(\Sigma_{x_0,T}))^2$, $\forall x_0 > 0$. *Furthermore, if, for a certain* $T > 0$, $(r,l) \in Y_T \times Y_T$ *is a local solution such that* $(r,l) \in (C_b^1(\Sigma_{x_0,T}))^2$, *then* $(r,l) \in (C_b^{1,1}(\Sigma_{x_0,T}))^2$. *Moreover, we have*

$$0 \leq -l \leq r \leq c, \quad \left\|\frac{l(.,t)}{x}\right\|_{L^\infty} \leq \left\|\frac{r(.,t)}{x}\right\|_{L^\infty} \leq \left\|\frac{r_0}{x}\right\|_{L^\infty}, \quad t \in [0,T'[\tag{1.10}$$

where $[0,T'[$ *is the maximal interval of local existence in Theorem 1.*

In the framework of Theorem 2, let us put (note that $r_0 = u_0 + a_0 \geq 0$, $l_0 = -u_0 + a_0 \leq 0$, $r_0^2 \geq l_0^2$):

$$c_0 = \left\|\frac{r_0}{x}\right\|_{L^\infty}. \tag{1.11}$$

With the technique of Lax (cf. [5]) we will prove the following blow-up result:

Theorem 3. *Under the hypothesis of Theorem 2, assume that* $c_0 > 1$ *and that there exists* $x_0 > 0$ *such that, in the interval* $[x_0, r_0(x_0)/c_0 + x_0]$, *we have*

$$r_{0x} < 0, \quad l_{0x} > 0, \quad \frac{5}{4}c_0 < |r_{0x}| < \frac{5}{4}c_0^2, \quad |l_{0x}| \geq c_0 e.$$

Then there exists $T' \in]0, 1/c_0]$ *such that*

$$\overline{\lim}_{t \to T'^-}(\|r\|_{Y_t} + \|l\|_{Y_t}) = +\infty. \tag{1.12}$$

Remark. The assumptions in Theorem 2 on the support of r_0 and l_0 can be replaced by some weaker hypothesis.

2. Local existence and smoothness of the solutions

We start with the proof of Theorem 1. Let us put

$$M_0 = \|(r_0, l_0)\|_{(C_b^{0,1})^2} = \|r_0\|_{L^\infty} + \|r_{0x}\|_{L^\infty} + \|l_0\|_{L^\infty} + \|l_{0x}\|_{L^\infty} \tag{2.1}$$

for two fixed $M > M_0$ and $T > 0$ let us consider the closed ball $B_{M,T}$ in $Y_T \times Y_Y$ centered in $(0, 0)$ and with radious M for the norm

$$\|(r, l)\| = \|r\|_{Y_T} + \|l\|_{Y_T}.$$

For $(v, w) \in B_{M,T}$ let us consider the linear system

$$\begin{cases} r_t + vr_x + \frac{v^2-w^2}{2x} = 0 \\ \\ l_t - wl_x + \frac{v^2-w^2}{2x} = 0 \end{cases} \tag{2.2}$$

with the initial data (1.6). For fixed $(x, t) \in Y_T$ let us consider the characteristic $X(\tau; x, t)$ passing in (x, t) defined by

$$\begin{cases} \frac{dX}{d\tau}(\tau; x, t) = v(X(\tau; x, t), \tau) \\ \\ X(t; x, t) = x \end{cases} \tag{2.3}$$

We can also define the characteristic

$$\begin{cases} \frac{d\widetilde{X}}{d\tau}(\tau; x, t) = -w(\widetilde{X}(\tau; x, t), \tau) \\ \\ \widetilde{X}(t; x, t) = x \end{cases} \tag{2.4}$$

Since $v(0, t) = w(0, t) \equiv 0$ by the hypothesis, the characteristics passing in a point $(0, t)$ are defined by the straight line $x = 0$. Denoting by $\dot{r}$ the derivative along the characteristic defined by (2.3) we can write the first equation of (2.2) as follows

$$\dot{r}(X(\tau; x, t), \tau) = -\frac{v^2 - w^2}{2x}(X(\tau; x, t), \tau)$$

and so

$$r(x,t) = r_0(X(0;x,t)) - \int_0^t \frac{v^2 - w^2}{2x}(X(\tau;x,t),\tau)\,d\tau. \tag{2.5}$$

We derive, for $t \leq T$,

$$\|r(.,t)\|_{L^\infty} \leq \|r_0\|_{L^\infty} + T\|(v,w)\|^2$$
$$\|r(.,t)\|_{L^\infty} \leq M_0 + TM^2. \tag{2.6}$$

and similarly, from the second equation in (2.2) and (2.4), we deduce, for $t \leq T$,

$$\|l(.,t)\|_{L^\infty} \leq M_0 + TM^2. \tag{2.7}$$

Now, if (x,t), $(\bar{x},\bar{t})$ are two points in $[0,+\infty[\times[0,T]$, $\bar{t} \leq t$, we have

$$r(x,t) - r(\bar{x},\bar{t}) = r(x,t) - r(\bar{x},t) + r(\bar{x},t) - r(\bar{x},\bar{t})$$

and

$$\begin{aligned} r(x,t) - r(\bar{x},t) &= r(X(t;x,t),t) - r(X(t;\bar{x},t),t) = \\ &= r_0(X(0;x,t)) - r_0(X(0;\bar{x},t)) - \\ &- \int_0^t \left[\frac{v^2 - w^2}{2x}(X(\tau;x,t),\tau) - \frac{v^2 - w^2}{2x}(X(\tau;\bar{x},t),\tau)\right] d\tau \end{aligned}$$

By well known properties of ordinary differential equations, we have, with $\bar{x}^*$ between x and $\bar{x}$,

$$\begin{aligned} |X(\tau;x,t) - X(\tau;\bar{x},t)| &\leq |x - \bar{x}| \left|\frac{\partial X}{\partial x}(\tau;\bar{x}^*,t)\right| \leq \\ &\leq |x - \bar{x}|\ exp \int_t^\tau \frac{\partial v}{\partial x}(X(s;\bar{x}^*,t),s)\,ds \leq \\ &\leq |x - \bar{x}| e^{TM}, \end{aligned}$$

and so

$$|r(x,t) - r(\bar{x},t)| \leq (M_0 + TM^2)e^{TM}|x - \bar{x}|. \tag{2.8}$$

We also have

$$\begin{aligned} r(\bar{x},t) - r(\bar{x},\bar{t}) &= r(X(t;\bar{x},t),t) - r(X(\bar{t};\bar{x},\bar{t}),\bar{t}) = \\ &= r_0(X(0;\bar{x},t)) - r_0(X(0;\bar{x},\bar{t})) - \int_0^t \frac{v^2 - w^2}{2x}(X(\tau;\bar{x},t),\tau)\,d\tau + \\ &+ \int_0^{\bar{t}} \frac{v^2 - w^2}{2x}(X(\tau;\bar{x},\bar{t}),\tau)\,d\tau \end{aligned}$$

and, with $\bar{t} \leq \bar{t}^* \leq t$,

$$|X(\tau;\bar{x},t) - X(\tau;\bar{x},\bar{t})| \leq |t-\bar{t}| \left|\frac{\partial X}{\partial t}(\tau;\bar{x},\bar{t}^*)\right| \leq$$
$$\leq |t-\bar{t}|\,|v(\bar{x},\bar{t}^*)| \quad \exp\int_{\bar{t}^*}^{\tau} \frac{\partial v}{\partial x}(X(s;\bar{x},\bar{t}^*),s)\,ds \leq$$
$$\leq |t-\bar{t}| M e^{TM}$$

Moreover, we have

$$\int_0^t \frac{v^2-w^2}{2x}(X(\tau;\bar{x},t),\tau)\,d\tau - \int_0^{\bar{t}} \frac{v^2-w^2}{2x}(X(\tau;\bar{x},\bar{t}),\tau)\,d\tau =$$
$$= \int_{\bar{t}}^t \frac{v^2-w^2}{2x}(X(\tau;\bar{x},t),\tau)\,d\tau +$$
$$+ \int_0^{\bar{t}} \left[\frac{v^2-w^2}{2x}(X(\tau;\bar{x},t),\tau) - \frac{v^2-w^2}{2x}(X(\tau;\bar{x},\bar{t}),\tau)\right] d\tau$$

Hence, we derive,

$$|r(\bar{x},t) - r(\bar{x},\bar{t})| \leq [(M_0 M + TM^3)e^{TM} + M^2]\,|t-\bar{t}|. \tag{2.9}$$

From (2.5), (2.6), (2.8) and (2.9) we deduce that $r \in Y_T$ and the same result can be proved for l. Moreover, there are M_1 and T_1 such that, if $M_0 < M \leq M_1$ and $T \leq T_1$, then for $(v,w) \in B_{M,T}$ we have

$$(r,l) \in B_{M,T}.$$

Following the ideas of [6, ch.1], and since $B_{M,T}$ is closed in

$$(C_b([0,\infty[\times[0,T_1]))^2,$$

to prove that, for fixed (r_0,l_0), the map $(v,w) \xrightarrow{J} (r,l)$ has a unique fixed point in $B_{M,T}$ it is enough to obtain the following estimate

$$\|J(v,w) - J(\bar{v},\bar{w})\|_{L^\infty} \leq \alpha\|(v,w)-(\bar{v},\bar{w})\|_{L^\infty} \tag{2.10}$$

for a certain $\alpha \in]0,1[$ and for all $(v,w),(\bar{v},\bar{w}) \in B_{M,T}$.

From the first equation in (2.2) for (v,w), (l,r) and $(\bar{v},\bar{w})$, $(\bar{r},\bar{l}) = J(\bar{v},\bar{w})$, we derive with $\tilde{r} = r - \bar{r}$ (cf. [6, ch.1] for a similar estimate),

$$\frac{\partial\tilde{r}}{\partial t} + v\frac{\partial\tilde{r}}{\partial x} = -(v-\bar{v})\frac{\partial\bar{r}}{\partial x} - \frac{v^2-w^2}{2x} + \frac{\bar{v}^2-\bar{w}^2}{2x}$$

and so, with $X(\tau; x, t)$ defined by (2.3), we obtain, by integrating and estimating,

$$\|\widetilde{r}(.,t)\|_{L^\infty} \le t\|v-\bar{v}\|_{L^\infty}\left(\|\bar{r}_x\|_{L^\infty} + \frac{1}{2}\|v_x\|_{L^\infty} + \frac{1}{2}\|\bar{v}_x\|_{L^\infty}\right) + \\ + \frac{t}{2}\|w-\bar{w}\|_{L^\infty}\big(\|w_x\|_{L^\infty} + \|\bar{w}_x\|_{L^\infty}\big),$$

$$\|\widetilde{r}(.,t)\|_{L^\infty} \le 2MT(\|v-\bar{v}\|_{L^\infty} + \|w-\bar{w}\|_{L^\infty})$$

and analogous estimate for $\|\widetilde{l}(.,t)\|_{L^\infty}$, with $\widetilde{l} = l - \bar{l}$, and this achieves the proof for "small" initial data (r_0, l_0), say $\|(r_0, l_0)\|_{\left(C_b^{0,1}\right)^2} \le M_0$.

Now, for a given initial data (r_0, l_0) let us choose $\lambda > 0$ such that $(\bar{r}_0, \bar{l}_0) = \lambda(r_0, l_0)$ verify $\|(\bar{r}_0, \bar{l}_0)\|_{\left(C_b^{0,1}\right)^2} \le M_0$, and let be $(\bar{r}, \bar{l})$ the unique solution in $Y_T \times Y_T$ of the corresponding Cauchy problem (1.5), (1.6). Let us put

$$r(x,t) = \frac{1}{\lambda}\bar{r}\,(x, t/\lambda)\,, \quad l(x,t) = \frac{1}{\lambda}\bar{l}\,(x, t/\lambda)\,.$$

We have

$$r(x,0) = r_0(x), \quad l(x,0) = l_0(x)$$

and (for $t \le \lambda T$):

$$r_t(x,t) + r r_x(x,t) + \frac{r^2 - l^2}{2x}(x,t) = \\ = \frac{1}{\lambda^2}\left(\bar{r}_t(x,t/\lambda) + \bar{r}\bar{r}_x(x,t/\lambda) + \frac{\bar{r}^2 - \bar{l}^2}{2x}(x,t/\lambda)\right) = 0$$

and also

$$l_t(x,t) + l l_x(x,t) + \frac{r^2 - l^2}{2x}(x,t) = 0$$

and the theorem is proved. □

To prove Theorem 2 we must introduce the approximate Cauchy problem

$$\begin{cases} r_{\varepsilon t} + r_\varepsilon r_{\varepsilon x} + \frac{r_\varepsilon^2 - l_\varepsilon^2}{2(x+\varepsilon)} = 0 \\ & x > 0, \ t \ge 0 \\ l_{\varepsilon t} - l_\varepsilon l_{\varepsilon x} + \frac{r_\varepsilon^2 - l_\varepsilon^2}{2(x+\varepsilon)} = 0 \end{cases} \tag{2.11}$$

with the same initial data given by (r_0, l_0). It is easy to see, by inspection of the proof of theorem 1, that the same proof applies to this regular case and moreover we can find a common (for $\varepsilon > 0$) interval $[0, T]$ of local existence of solution for the Cauchy problem with T depending only on the norm $\|(r_0, l_0)\|_{\left(C_b^{0,1}\right)^2}$ of the initial data. Furthermore, we have the estimate

$$\|(r_\varepsilon, l_\varepsilon)\|_{Y_T \times Y_Y} \leq c_1, \quad \forall \varepsilon \geq 0 \tag{2.12}$$

with c_1 only depending on $\|(r_0, l_0)\|_{\left(C_b^{0,1}\right)^2}$. Moreover, if $(r_0, l_0) \in \left(C_b^{1,1}\right)^2$ we also have, for $\varepsilon > 0$,

$$(r_\varepsilon, l_\varepsilon) \in \left(C_b^{1,1}([0, +\infty[\times[0, T])\right)^2$$

(cf. theo. 4.3 in ch.2 of [6]). Finally, under the hypothesis of Theorem 2 it can be proved, as we have made in [2] for the singular case by applying the vanishing viscosity method and an uniqueness theorem of Kruzkov's type, that

$$\begin{gathered} 0 \leq -l_\varepsilon \leq r_\varepsilon \leq c_1 \quad \text{in} \quad [0, +\infty[\times[0, T], \\ \left\|\frac{l_\varepsilon(.,t)}{x+\varepsilon}\right\|_{L^\infty} \leq \left\|\frac{r_\varepsilon(.,t)}{x+\varepsilon}\right\|_{L^\infty} \leq \left\|\frac{r_0}{x}\right\|_{L^\infty}, \quad t \in [0, T]. \end{gathered} \tag{2.13}$$

If we obtain a proof of the equicontinuity, in $\Sigma_{x_0,T}$, for a fixed $x_0 > 0$, of the first derivatives of the sequence $(r_\varepsilon, l_\varepsilon)$ we can apply Ascoli's theorem in order to obtain a subsequence, yet denoted by $(r_\varepsilon, l_\varepsilon)$, converging in

$$(C_b([0, +\infty[\times[0, T]))^2 \cap \left(C_b^1(\Sigma_{x_0,T})\right)^2$$

for a weak entropy solution $(\bar{r}, \bar{l})$ for the Cauchy problem (1.5), (1.6) (see [2] for the definition) such that

$$0 \leq -l \leq r \leq c_1 \quad \text{and} \quad \left\|\frac{l(.,t)}{x}\right\|_{L^\infty} \leq \left\|\frac{r(.,t)}{x}\right\|_{L^\infty} \leq \left\|\frac{r_0}{x}\right\|_{L^\infty} \tag{2.14}$$

By the uniqueness theorem proved in [2], we derive $(\bar{r}, \bar{l}) = (r, l)$, the solution found in Theorem 1, and the estimates (2.14) hold for $t \in [0, T'[$, maximal interval of local existence in Theorem 1 (cf.[2], theo. 2).

Now we pass to the proof of the equicontinuity of the first derivatives, $p_\varepsilon = r_{\varepsilon x}$, $q_\varepsilon = r_{\varepsilon t}$, $\widetilde{p}_\varepsilon = l_{\varepsilon x}$, $\widetilde{q}_\varepsilon = l_{\varepsilon t}$. With the notation introduced in (2.3), (2.4) with $v = r_\varepsilon \geq 0$, $-w = -l_\varepsilon \geq 0$ (note that $c_1 \geq r_\varepsilon(x,t) \geq -l_\varepsilon(x,t) \geq 0$), by (1.5) we can write in a point $(x,t) \in \Sigma_{x_0,T}$ (droping the ε for simplicity):

$$\dot{p} = p_t + r p_x = -p^2 - \frac{rp - r\widetilde{p}}{x+\varepsilon} + \frac{r^2 - l^2}{2(x+\varepsilon)^2}$$

and so, with $p_0(x) = p(x, 0) = r_{0x}(x)$, and following the characteristic

$$p(x,t) = p(X(t;x,t),t) = p_0(X(0;x,t)) - \int_0^t p^2(X(\tau;x,t),\tau)\,d\tau - \int_0^t \frac{rp - l\widetilde{p}}{x+\varepsilon}(X(\tau;x,t),\tau)\,d\tau + \int_0^t \frac{r^2 - l^2}{2(x+\varepsilon)^2}(X(\tau;x,t),\tau)\,d\tau.$$

Hence, a.e. on $(x,t) \in \Sigma_{x_0,T}$,

$$p_x(x,t) = p_{0x}(X(0;x,t))\,\frac{\partial X}{\partial x}(0;x,t) -$$

$$- \int_0^t 2\,p\,p_x(X(\tau;x,t),\tau)\,\frac{\partial X}{\partial x}(\tau;x,t)\,d\tau -$$

$$- \int_0^t \frac{p^2 + rp_x - \widetilde{p}^2 - l\widetilde{p}_x}{x+\varepsilon}(X(\tau;x,t),\tau)\,\frac{\partial X}{\partial x}(\tau;x,t)\,d\tau +$$

$$+ \int_0^t \frac{rp - l\widetilde{p}}{(x+\varepsilon)^2}(X(\tau;x,t),\tau)\,\frac{\partial X}{\partial x}(\tau;x,t)\,d\tau +$$

$$+ \int_0^t \frac{rp - l\widetilde{p}}{(x+\varepsilon)^2}(X(\tau;x,t),\tau)\,\frac{\partial X}{\partial x}(\tau;x,t)\,d\tau -$$

$$- \int_0^t \frac{r^2 - l^2}{(x+\varepsilon)^3}(X(\tau;x,t),\tau)\,\frac{\partial X}{\partial x}(\tau;x,t)\,d\tau.$$

We point out that $x \geq x_0$ in $\Sigma_{x_0,T}$ and, by (2.12),

$$\left|\frac{\partial X}{\partial x}(\tau;x,t)\right| \leq \exp\int_\tau^t |p(X(s;x,t)|\,ds \leq e^{c_1 t} \leq e^{c_1 T},$$

with c_1 not depending on ε. Hence, by (2.12), we derive, with

$$f_\varepsilon(\tau) = \sup_x |p_{\varepsilon x}(x,\tau)| \quad \text{and} \quad \widetilde{f}_\varepsilon(\tau) = \sup_x |\widetilde{p}_{\varepsilon x}(x,\tau)|,$$

$$|p_{\varepsilon x}(X(t;x,t),t)| \leq (c_1 + c_1 T)e^{c_1 T} + c_1 e^{c_1 T}\int_0^t \left(f_\varepsilon(\tau) + \widetilde{f}_\varepsilon(\tau)\right)\,d\tau.$$

Similarly, we get, following the characteristic $\widetilde{X}(\tau; x, t)$:

$$|\widetilde{p}_{\varepsilon x}(\widetilde{X}(t; x, t), t)| \leq (c_1 + c_1 T)e^{c_1 T} + c_1 e^{c_1 T} \int_0^t \left(f_\varepsilon(\tau) + \widetilde{f}_\varepsilon(\tau)\right) d\tau.$$

Hence,

$$f_\varepsilon(t) + \widetilde{f}_\varepsilon(t) \leq (c_1 + c_1 T)e^{c_1 T} + c_1 e^{c_1 T} \int_0^t \left(f_\varepsilon(\tau) + \widetilde{f}_\varepsilon(\tau)\right) d\tau.$$

By Gronwall's inequality we derive

$$f_\varepsilon(t) + \widetilde{f}_\varepsilon(t) \leq (c_1 + c_1 T)\, e^{c_1 T(1+e^{c_1 T})} = c_2. \tag{2.15}$$

For $p_{\varepsilon t}$ and $\widetilde{p}_{\varepsilon t}$ we can derive a similar estimate.

Now, for $q = r_{\varepsilon t}$ we derive from (1.5) (always droping the ε for simplicity):

$$\dot{q} = q_t + r q_x = -qp - \frac{rq - r\widetilde{q}}{x + \varepsilon}$$

and so with $q_0(x) = q(x, 0) = -r_0 r_{0x}(x) - \dfrac{r_0^2 - l_0^2}{2(x+\varepsilon)}(x)$,

$$\begin{aligned} q(x, t) = & q(X(t; x, t), t) = \left(-r_0 r_{0x} - \frac{r_0^2 - l_0^2}{2(x+\varepsilon)}\right)(X(0; x, t)) - \\ & - \int_0^t qp(X(\tau; x, t), \tau)\, d\tau - \int_0^t \frac{rq - l\widetilde{q}}{x + \varepsilon}(X(\tau; x, t), \tau)\, d\tau. \end{aligned}$$

Hence, a.e. on $(x, t) \in \Sigma_{x_0, T}$,

$$\begin{aligned} & q_x(x, t) = \\ & = \left[-r_{0x}^2 - r_0 r_{0xx} - \frac{r_0 r_{0x} - l_0 l_{0x}}{x + \varepsilon} + \frac{r_0^2 - l_0^2}{2(x+\varepsilon)^2}\right](X(0; x, 0)) \cdot \frac{\partial X}{\partial x}(0; x, t) \; - \\ & \quad - \int_0^t (q_x p + q p_x)(X(\tau; x, t), \tau)\, \frac{\partial X}{\partial x}(\tau; x, t)\, d\tau \; - \\ & \quad - \int_0^t \frac{pq + rq_x - \widetilde{p}\widetilde{q} - l\widetilde{q}_x}{x + \varepsilon}(X(\tau; x, t), \tau)\, \frac{\partial X}{\partial x}(\tau; x, t)\, d\tau \; + \\ & \quad + \int_0^t \frac{rq - l\widetilde{q}}{(x+\varepsilon)^2}(X(\tau; x, t), \tau)\, \partial X \partial x(\tau; x, t)\, d\tau. \end{aligned}$$

With $g_\varepsilon(\tau) = \sup_x |q_{\varepsilon x}(x,\tau)|$ and $\widetilde{g}_\varepsilon(\tau) = \sup_x |\widetilde{q}_{\varepsilon x}(x,\tau)|$, we derive

$$|q_{\varepsilon x}(x,t)| \leq [c_1 + (c_1+c_2)T]\, e^{c_1 T} + c_1 e^{c_1 T} \int_0^t (g_\varepsilon(\tau) + \widetilde{g}_\varepsilon(\tau))\, d\tau$$

and a similar estimate for $|\widetilde{q}_{\varepsilon x}(x,t)|$. Hence, by (2.15),

$$g_\varepsilon(t) + \widetilde{g}_\varepsilon(t) \leq [c_1 + (c_1+c_2)T]\, e^{c_1 T} + c_1 e^{c_1 T} \int_0^t (g_\varepsilon(\tau) + \widetilde{g}_\varepsilon(\tau))\, d\tau.$$

By Gronwall's inequality we deduce

$$g_\varepsilon(t) + \widetilde{g}_\varepsilon(t) \leq [c_1 + (c_1+c_2)T]\, e^{c_1 T(1+e^{c_1 T})}.$$

For $q_{\varepsilon t}$ and $\widetilde{q}_{\varepsilon t}$ we can derive a similar estimate. Hence, we have obtained a uniform (in $\varepsilon > 0$) estimate in $\left(C_b^{1,1}(\Sigma_{x_0,T})\right)^2$ for $(r_\varepsilon, l_\varepsilon)$. We derive $(r,l) \in \left(C_b^1(\Sigma_{x_0,T})\right)^2$ but we even obtain $(r,l) \in \left(C_b^{1,1}(\Sigma_{x_0,T})\right)^2$ since the previous estimates are uniform. More generally, under the assumptions of theorem 2, if $(r,l) \in (Y_T \times Y_T) \cap \left(C_b^1(\Sigma_{x_0,T})\right)^2$, for a fixed $x_0 > 0$, is a local solution of (1.5), (1.6), we can prove, by estimating

$$p(x,t) - p(\bar{x},\bar{t}),\ q(x,t) - q(\bar{x},\bar{t}),\ \widetilde{p}(x,t) - \widetilde{p}(\bar{x},\bar{t}),\ \widetilde{q}(x,t) - \widetilde{q}(\bar{x},\bar{t}),$$

where $p = r_x$, $q = r_t$, $\widetilde{p} = l_x$, $\widetilde{q} = l_t$, that $(r,l) \in \left(C_b^{1,1}(\Sigma_{x_0,T})\right)^2$ (cf. theo. 3.1 in Ch.1 of [6]). □

3. Blow-up of some solutions

In this section we will prove Theorem 3. Under the hypothesis of this theorem let $[0, T'[$ be the maximal interval of existence of a local solution $(r,l) \in (Y_T \times Y_T) \cap \left(C_b^1(\Sigma_{x_0,T})\right)^2$, $\forall T < T'$, for a fixed $x_0 > 0$. By Theorem 2 we have $(r,l) \in (Y_T \times Y_T) \cap \left(C_b^{1,1}(\Sigma_{x_0,T})\right)^2$, $\forall T < T'$. Let us suppose $T' > 1/c_0$. For $p = -l_x$ we derive from (1.5), if $p'(\tau)$ is the derivative (which exists a.e. on τ) along the characteristic defined by

$$\begin{cases} \frac{d}{d\tau}\widetilde{X}(\tau;\widetilde{x}_0,0) = -l\,(\widetilde{X}(\tau;\widetilde{x}_0,0),\tau), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq \tau \leq 1/c_0 \\ \widetilde{X}(0;\widetilde{x}_0,0) = \widetilde{x}_0 = r_0(x_0)/c_0 + x_0, \end{cases} \tag{3.1}$$

$$p'(\tau) + p^2(\tau) - \frac{l}{x}(\tau)p(\tau) - \frac{rr_x}{x}(\tau) + \frac{r^2 - l^2}{2x^2}(\tau) = 0,$$

with $p(\tau) = p(\widetilde{X}(\tau; \widetilde{x}_0, 0), \tau)$. Hence, assuming $q = r_x \leq 0$ along the characteristic for $\tau \leq 1/c_0$, we derive since $r \geq 0,\ r^2 - l^2 \geq 0$,

$$\begin{cases} p'(\tau) + p^2(\tau) - \frac{l}{x}(\tau)p(\tau) \leq 0, \\ & 0 \leq \tau \leq 1/c_0 \\ p(0) = -l_{0x}(\widetilde{x}_0) < 0. \end{cases} \tag{3.2}$$

Putting $v(\tau) = e^{h(\tau)}p(\tau),\ h(\tau) = \int_0^\tau [-(l/x)(s)]\,ds$ (recall that $-l/x \geq 0$ and so $h'(\tau) \geq 0$), we deduce

$$\begin{cases} v'(\tau) + e^{-h(\tau)}v^2(\tau) \leq 0, \\ & 0 \leq \tau \leq 1/c_0 \\ v(0) = p(0) < 0. \end{cases}$$

Now, following an idea of Lax (cf. [5]), we compare v with θ solution of the Cauchy problem

$$\begin{cases} \theta'(\tau) + e^{-h(\tau)}\theta^2(\tau) = 0, \\ & 0 \leq \tau \leq 1/c_0 \\ \theta(0) = v(0) = p(0) < 0. \end{cases}$$

We derive

$$v(\tau) \leq \theta(\tau) = p(0)\left[1 + p(0)\int_0^\tau e^{-h(s)}ds\right]^{-1}.$$

But we have $\int_0^\tau e^{-h(s)}ds \geq \tau e^{-h(\tau)}$ and

$$|h(\tau)| \leq \tau \sup_{0 \leq s \leq \tau} \left\| \frac{l(.,s)}{x} \right\|_{L^\infty} \leq \tau \left\| \frac{r_0}{x} \right\|_{L^\infty} = \tau c_0$$

and so $\int_0^\tau e^{-h(s)}ds \geq \tau e^{-c_0\tau}$. The function $\tau e^{-c_0\tau}$ increases till $\tau = 1/c_0$. Since $p(0) < 0,\ |p_0| \geq c_0\, e$, we derive

$$1 + p(0)\int_0^\tau e^{-h(s)}ds \leq 1 + p(0)\tau e^{-c_0\tau} = 0$$

for a certain $T_1 \leq 1/c_0$. Hence, $\underset{\tau \to T_1^-}{\underline{lim}}\ v(\tau) \leq \underset{\tau \to T_1^-}{\underline{lim}}\ \theta(\tau) = -\infty$. We conclude

$$\underset{\tau \to T_1^-}{\underline{lim}}\ [-l_x(\widetilde{X}(\tau; \widetilde{x}_0, 0), \tau)] = -\infty$$

and the solution blows-up in $[0, +\infty[\times[0, 1/c_0]$ which is absurd.

Now we need to prove that $q = r_x \leq 0$ on the considered characteristic (see fig.1) for $\tau \leq 1/c_0$ (remember that we have assumed $T' > 1/c_0$). Let us consider the family of characteristics defined by

$$\begin{cases} \frac{d}{d\tau} X(\tau; \bar{x}_0, 0) = r(X(\tau; \bar{x}_0, 0), \tau), \\ & 0 \leq \tau \leq 1/c_0 \\ X(0; \bar{x}_0, 0) = \bar{x}_0 \in [x_0, \widetilde{x}_0] \end{cases} \tag{3.3}$$

such that they cross the characteristic defined by (3.1). For each P belonging to the characteristic defined by (3.1) ($\tau \leq 1/c_0$), there is one characteristic of type (3.3) passing in P.

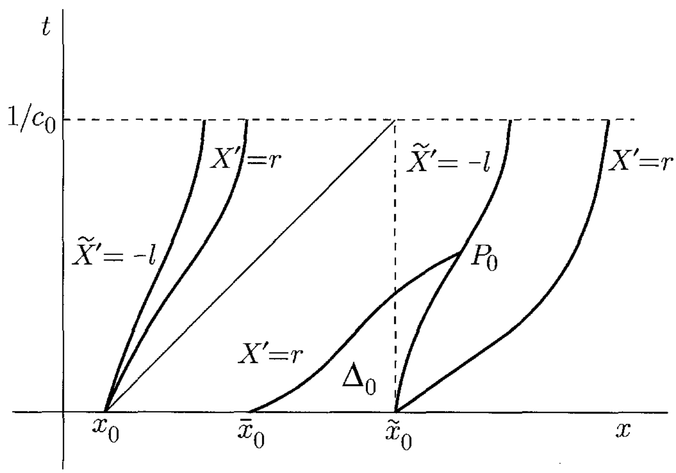

Figure 1

If $q = r_x$ we denote by $\dot{q}(\tau)$ the derivative of q along this characteristic ($\dot{q}$ exists a.e. on τ). We derive from (1.5)

$$\begin{cases} \dot{q}(\tau) + q^2(\tau) + \frac{rq - ll_x}{x}(\tau) - \frac{r^2 - l^2}{2x^2}(\tau) = 0 \\ q(0) = \bar{x}_0 \in [x_0, \widetilde{x}_0]. \end{cases} \tag{3.4}$$

If we suppose $p = -l_x \leq 0$ on the characteristic defined by (3.3) till its intersection (at the time $T_{\bar{x}_0} \leq 1/c_0$) with the characteristic defined by (3.1) we derive from (3.2) and for each $\delta \in]0, 1[$ (recall that $r^2 - l^2 \geq 0$):

$$\dot{q}(\tau) \leq (-1+\delta)q^2(\tau) + c(\delta), \qquad \tau \in [0, T_{\bar{x}_0}],$$

where $c(\delta) = \frac{1}{\delta}\frac{5}{4}c_0^2$ (note that $|q(0)| < \frac{5}{4}c_0^2$). Since $(1-\delta)q^2 - c(\delta) < 0$, we derive

$$\frac{dq}{(1-\delta)q^2 - c(\delta)} \geq -dt,$$

that is, with

$$K_1(\delta) = 2\sqrt{(1-\delta)c(\delta)}, \quad K(\delta) = \sqrt{c(\delta)/(1-\delta)}\ ,$$

and by integration between 0 and $\tau < T_{\bar{x}_0}$,

$$\frac{1}{K_1(\delta)}\left[\log\left|\frac{q(\tau) - K(\delta)}{q(\tau) + K(\delta)}\right|\right]_{q_0}^{q(\tau)} \geq -\tau.$$

We deduce

$$\frac{K(\delta) - q(\tau)}{q(\tau) + K(\delta)} \geq \frac{K(\delta) - q_0}{q_0 + K(\delta)}\, e^{-K_1(\delta)\tau} = f_\delta(\tau)$$

and so

$$q(\delta) \leq K(\delta)\,\frac{1 - f_\delta(\tau)}{1 + f_\delta(\tau)}. \tag{3.5}$$

We want to choose $\delta \in]0, 1[$ such that for $\tau \leq 1/c_0$, $f_\delta(\tau) \geq f_\delta(1/c_0) > 1$. We have

$$f_\delta(1/c_0) > 1 \iff \sqrt{\frac{c(\delta)}{q_0^2(1-\delta)}}\left[1 - e^{-2/c_0\sqrt{(1-\delta)c(\delta)}}\right] > -1 - e^{-2/c_0\sqrt{(1-\delta)c(\delta)}}$$

where $c(\delta) = \frac{1}{\delta}\frac{5}{4}c_0^2$. When $\delta \to 1^-$, the right hand side of the previous inequality converges to -2 and the left hand side to $-5/2\ c_0\,|q_0|^{-1}$. Hence, we can choose a δ if $|q_0| > 5/4\ c_0$ as in the hypothesis of the theorem 3. From (3.5) we derive $q(\tau) < 0$ till the characteristic cross the characteristic defined by (3.1). Now, since $q_0 = r_{0x} < 0$, $p_0 = -l_{0x} < 0$ in $[x_0, \widetilde{x}_0]$, there exists a closed "triangle" Δ_0 with one vertex in $(\widetilde{x}_0, 0)$, one side $[\bar{x}_0, \widetilde{x}_0] \times \{0\}$, where $\bar{x}_0 \in]x_0, \widetilde{x}_0[$, the second side being the part of the characteristic defined by (3.1) between $(\widetilde{x}_0, 0)$ and the intersection P_0 of the characteristic of type (3.3) starting in $(\bar{x}_0, 0)$ and the third side being the part of this characteristic between $(\bar{x}_0, 0)$ and P_0 (cf. fig.1), such that $q = r_x < 0$ and $p = -l_x < 0$ in Δ_0. Let Δ be the maximal of the triangles of this type and suppose that $\bar{\Delta}$ does not contains the characteristic defined by (3.1) (for $0 \leq \tau \leq 1/c_0$). Since $p \leq 0$ in the side of $\bar{\Delta}$ being a part of a characteristic of type (3.3), we deduce, as in the second part of the proof, that $q < 0$ on this characteristic. Finally, for a point P (not on the x-axis) on this side, let us consider the backward characteristic of type (3.1) passing in the point P: this characteristic lies in $\bar{\Delta}$. Since $q < 0$, we can apply the first part of the proof and we derive $p(P) < 0$. Hence Δ is not maximal. Therefore $q \leq 0$ in the characteristic defined by (3.1) (for $0 \leq \tau \leq 1/c_0$) and the blow-up result follows. □

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